2

Leading from Strength: Playing Dominant Strategies

2.1 Introduction

This chapter shows clearly that elementary game theory is extremely insightful in analyzing real-world problems. The challenge for the student in dealing with the material in this chapter is that of *thinking systematically and logically*—handwaving should not be tolerated. Even highly sophisticated students can fail to understand the logic of eliminating dominated strategies in the *Prisoner's Dilemma* (§2.6), the *Second Price Auction* (§2.8), and the *Centipede Game* (§2.18), although the reasoning is completely elementary. I present these models in class, with answers and extensive discussion, as well as *Hagar's Battles* (§2.10) because it illustrates clear thinking, and *Strategic Voting* (§2.14) because it is a nice example from political theory. The problems are mostly straightforward, but *Poker with Bluffing* (§2.17) should be extra credit for computer hackers.

2.2 Dominant Strategies

A powerful way of finding Nash equilibria of games is to eliminate what are called *dominated strategies*. Suppose s_i and s_i' are two strategies for player i in a normal form game. We say s_i' is *strictly dominated* by s_i if, for every choice of strategies of the other players, i's payoff from choosing s_i is *strictly greater* than i's payoff from choosing s_i'. We say s_i' is *weakly dominated* by s_i if, for every choice of strategies of the other players, i's payoff from choosing s_i is *at least as great* as i's payoff from choosing s_i'.

It is clear that a strictly dominated strategy s_i is never part of a Nash equilibrium, since the player i could do better by switching to a strategy that dominates s_i. We can therefore can eliminate such strategies

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without losing any Nash equilibria. We call this process the *elimination of strictly dominated strategies*.

After we have eliminated strictly dominated strategies of one player, we may find that there are now strictly dominated strategies for another player, which we can eliminate as well. We can continue this until there are no strictly dominated strategies left. We call this the *iterated elimination of strictly dominated strategies*.

What about eliminating weakly dominated strategies? If the iterated elimination of weakly dominated strategies leaves exactly one strategy for each player, the resulting strategy profile is a Nash equilibrium (§2.5). However some Nash equilibria can be discarded this way. In particular, a player may use a weakly dominated strategy in a Nash equilibrium in, in the extensive form of the game, the resulting equilibrium does not reach those parts of the game tree where the strategy is worse than some alternative (§2.5). On the other hand, a Nash equilibrium that uses weakly dominated strategies has very poor dynamic properties: it cannot be evolutionarily stable (§7.2), it disappears when even an infinitesimal probability of player error is assumed (§5.16), and it is not stable under replicator and related dynamics (§9.2). Moreover, there always exists a Nash equilibrium in a finite game that does *not* use weakly dominated strategies.[1] For these reasons, we will always feel free to eliminate weakly dominated strategies unless there is a reason for not so doing. The interested reader can refer to the discussions in van Damme (1991) and Weibull (1995) for more ample treatments of this issue.

2.3 Backward Induction: Pruning the Game Tree

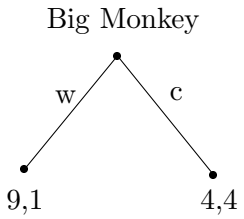
We can also eliminate dominated strategies in extensive form games with *complete information* (i.e., where each information set is a single node). Choose any terminal node $t \in T$ and find the parent node of this terminal node, say node a, suppose player i chooses at a, and suppose i's highest payoff at a is attained at node $t' \in T$. Erase all the branches from a, so a becomes a terminal node, and attach the payoffs from t' to the new terminal node a. Also record i's move at a, so you can specify i's equilibrium strategy when your are finished the analysis. Repeat for all terminal nodes of the original game. When you are done, you will have an extensive form game that is one less level 'deep' than the original game. Now repeat the process as many times as is necessary,

[1] You can easily prove this, but you must use the fact that every finite game has a Nash equilibrium (§4.3).

until the game tree has just one node (except for Nature's moves, if any). Reassembling the moves you have recorded for each player, you will have a Nash equilibrium.

We call this formally *backward induction*, because we start at the end of the game and move backward. We also call it more informally *pruning the game tree* because we successively lop off branches of the game tree, much as a gardener prunes a real tree. Notice that backward induction eliminates weakly dominated strategies, but cannot easily distinguish between strongly and weakly dominated strategies, since it never looks at the whole game tree. Therefore backward induction can eliminate Nash equilibria that use weakly dominated strategies.

For an example of pruning the game tree, consider Figure 1.1, the Big Monkey and Little Monkey game where Big Monkey goes first. We start with the terminal node labeled (0,0), and follow it back to the Little Monkey node on the left. At this node w is strictly dominated by c since $1 > 0$, so we write down that Little Monkey chooses c there, erase the two branches, and label the new terminal node (9,1). We locate the next terminal node in the original game tree, (4,4), and following back to the Little Monkey node on the right. At this node c is strictly dominated by w, so we write down that Little Monkey chooses w there, erase the two branches, and label the new terminal node (4,4). This gives the following:



Now we prune this smaller game tree—this time of course it's trivial. We find the first terminal node, (9,1), which leads back to Big Monkey. Here c is strictly dominated, so we erase the two branches, record that Big Monkey chooses w, and write (9,1) under the one remaining node. We now have our solution: Big Monkey chooses w, Little Monkey chooses cw, and the payoffs are (9,1).

2.4 Eliminating Dominated Strategies

(a) Solve the normal form game on left of Figure 2.1 by eliminating dominated strategies. Verify that the resulting solution is a Nash equilibrium of the game.

(b) Can a Nash equilibrium to the game on the right of Figure 2.1 be found by the iterated elimination of dominated strategies? If so, describe exactly in what order you delete strategies.

	N	C	J
N	73,25	57,42	66,32
C	80,26	35,12	32,54
J	28,27	63,31	54,29

	a	b	c	d	e
A	63,-1	28,-1	-2,0	-2,45	-3,19
B	32,1	2,2	2,5	33,0	2,3
C	54,1	95,-1	0,2	4,-1	0,4
D	1,-33	-3,43	-1,39	1,-12	-1,17
E	-22,0	1,-13	-1,88	-2,-57	-3,72

Figure 2.1: Eliminating Dominated Strategies

2.5 Dominated Strategies: Concepts and Definitions

(a) Describe *backward induction* in an extensive form game.

(b) Define the notion of a strategy being *strictly dominated* in a normal or extensive form game. Explain in words why a strictly dominated strategy can never be used by a player in a Nash equilibrium.

(c) Describe the *iterated elimination of strictly dominated strategies* and explain why a strategy eliminated in this manner cannot be used by a player in a Nash equilibrium.

(d) Explain why, if the iterated elimination of weakly dominated strategies leaves exactly one strategy for each player, the resulting strategy profile is a Nash equilibrium.

(e) Unlike the case of strongly dominated strategies, when you eliminate weakly dominated strategies, you can throw away Nash equilibria. Show this by finding two Nash equilibria in the diagram below.

	c1	c2	c3
r1	1,0	-2,-1	0,1
r2	1,2	-5,-1	0,0

(f) Suppose strategy s_i is weakly dominated by s_i' for player i in an extensive form game, and suppose $s = (s_1, \ldots, s_n)$ is a Nash equilibrium for the game. Let a be a node where i chooses and s_i' gives i a higher payoff that s_i' for some strategy profile of the other players. Explain why no path through the game tree compatible with s passes through node a. Hint: remember our assumption that branches representing Nature's moves all have strictly positive probability, and refer to (1.1).

2.6 The Prisoner's Dilemma

Two partners in a law firm are accused of double billing their clients. The Independent Counsel has enough evidence to send them to jail for one year each. However, if one of the law partners confesses and provides incriminating evidence on the other, he will go free and his partner will go to jail for ten years. If both confess, however, they will each get eight years in jail—ten years for the crime minus two years for confessing.

(a) Draw a game tree for the problem, labelling the nodes appropriately. You may assume the partners move simultaneously;

(b) Show that there is a unique Nash equilibrium, obtained by eliminating dominated strategies;

(c) Draw the normal form game for the problem, and use it to come to the same conclusion;

(d) Suppose one law partner goes first and the other sees the move. Draw the normal form game now and find the unique Nash equilibrium by eliminating dominated strategies.

2.7 An Armaments Game

A fighter command has four strategies, and its opponent bomber command has three counter-strategies. The diagram below shows the probability the fighter destroys the bomber. Use the elimination of dominated strategies to determine a solution to this game.

		Bomber Command: Full fire, low speed	Bomber Command: Partial fire, medium speed	Bomber Command: No fire, high speed
Fighter Command	Guns	0.30	0.25	0.15
	Rockets	0.18	0.14	0.16
	Toss-bombs	0.35	0.22	0.17
	Ramming	0.21	0.16	0.10

2.8 Second Price Auction

A single object is to be sold at auction. There are $n > 1$ bidders, each submitting a single bid, in secret, to the seller. The value of the object to bidder i is v_i. The winner of the object is the highest bidder, but i pays only the next highest bid.

(a) Show that 'truth-telling' (i.e., each player i bids v_i) is a dominant strategy for each player;

(b) Does your analysis depend on whether or not others tell the truth?

(c) What are some reasons that real second price auctions might not conform to the assumptions of this model?

2.9 The Landlord and the Eviction Notice

A landlord has three tenants, Mr. A, Ms. B, and Msgr. C, in a rent-controlled apartment building in New York. A new law says that the landlord has the right to evict one tenant per building. The landlord calculates that the value of a vacant apartment is \$15,000, both to the tenant and to her. She sends the following letter to each of the tenants: "Tomorrow I will be visit your building. I will offer Mr. A \$1,000 if he agrees to vacate his apartment voluntarily; otherwise I will evict him. If Mr. A agrees to vacate voluntarily, I will then offer Ms. B \$1,000, and if she refuses, I will evict her. If she accepts, I will evict Msgr. C."

(a) Write a game tree for this situation, and find the dominant strategy equilibrium.

(b) What if there were 10 tenants instead of three?

(c) What if the landlord offered only \$10 if a tenant leaves voluntarily?

2.10 Hagar's Battles

There are ten battlefields with military values $a_1 < \ldots < a_{10}$. Each player is endowed with $n_i < 10$ soldiers ($i = 1, 2$). A player's strategy is a decision to send his soldiers to these various battlefields. A player can send at most one soldier to a given battlefield. When the fighting begins, each player wins a_j for each battlefield where he has a soldier but his opponent does not. The winner of the war is the army whose occupied territory has the highest total military value. Show that this game has a unique dominant strategy equilibrium.

2.11 An Increasing Bid Auction

A Ming vase is to be sold at auction. There are n bidders, and the value of the vase to bidder i is $v_i > 0$, $i = 1, \ldots, n$. The auctioneer begins the bidding at zero, and raises the price at the rate of \$1 per second. All bidders who are willing to buy the vase at the stated price put their hands up simultaneously. This continues until there is only one arm raised. This last and highest bidder is the winner of the auction and must buy the vase for the stated price. Show that this auction has the same optimal strategies and the same outcome as the Second Price Auction. (Hint: First show that the only undominated pure strategies for bidder i take the form of choosing a price p_i and keeping a hand up until the auctioneer's price goes higher than this. Then find the optimal p_i.)

2.12 Football Strategy

In a football game, the offense has two strategies, Run or Pass. The defense also has three strategies, Counter Run, Counter Pass, or Blitz the quarterback. After studying many games, the statisticians came up with the following table, giving the expected number of yards game when the various strategies in the figure are followed. Use the elimination of dominated strategies to find a solution to this game, which is depicted in Figure 2.2.

2.13 A Military Strategy Game

Country A and country I are at war. The two countries are separated by a series of rivers, illustrated in Figure 2.3.

Offense	Defense: Counter Run	Defense: Counter Pass	Defense: Blitz
Run	3,-3	7,-7	15,-15
Pass	9,-9	8,-8	10,-10

Figure 2.2: Football Strategy

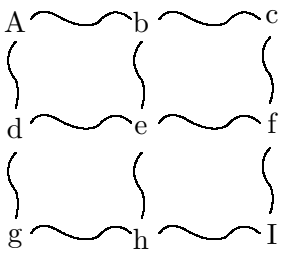


Figure 2.3: A Military Strategy Game

Country I sends a naval fleet with just enough supplies to reach A. The fleet must stop for the night at intersections (e.g., if the fleet takes the path IhebA, it must stop the first night at h, the second at e, and the third at b). If unhindered, on the fourth day the fleet will reach A and destroy Country A. Country A can send a fleet to prevent this. Country A's fleet has enough supplies to visit three contiguous intersections, starting from A (e.g., Abcf). If it catches Country I's fleet (i.e., if both countries stop for the night at the same intersection), it destroys the fleet and wins the war. List the strategies of the two countries, and make a payoff matrix for these strategies, assuming the winner gets 1 and the loser -1. Eliminate dominated strategies to find the set of undominated strategies for the two countries.

2.14 Strategic Voting

Three legislators are voting on whether to give themselves a pay raise. The raise is worth b, but each legislator who votes for the raise incurs a cost of voter resentment equal to $c < b$. The outcome is decided by majority rule.

(a) Draw a game tree for the problem, assuming the legislators vote sequentially and publicly (i.e., the second voter sees the first's vote, and the third see the votes of the other two).

(b) Find a Nash equilibrium for this game by using backward induction. Show that it is best to go first?

(c) Show that there is a Nash equilibrium in which the third legislator votes no, whatever the other players do, and this equilibrium favors the third legislator. Why can't this equilibrium be found by backward induction? Is there something strange about this Nash equilibrium?

2.15 The Debtor and His Creditors

A debtor owes \$15,000 to each of two creditors but he only has \$25,000. If he defaults on the debt, he will lose the whole amount and the legal costs of filing bankruptcy and litigating the liquidation of his assets will be \$15,000, so each of the debtors will collect \$5,000. The debtor has his lawyer draw up the following letter, which he sends to each of the creditors: "I hereby offer you \$5,001 if both you and my other creditor agree to cancel my debt. If either or both of you decline this offer, I will be legally in default."

Write a game tree for this situation, and show that it is a dominant strategy for each creditor is to accept the offer, allowing the debtor to eradicate the debt and retain the amount \$14,998 for himself. What does this tell you about bankruptcy law?

2.16 Eliminating Dominated Strategies *ad Absurdum*

Consider an n-player game in which each player announces simultaneously an integer between 1 and 1000. The winner is the player whose announcement is closest to 1/2 of the average of all the announcements. In the case of a tie, the prize is given by random draw among the winners.

(a) Show that if players iterate the elimination of dominated strategies, the only remaining strategy is to announce 1.

(b) To most people this does not sound like what would actually happen if n people played this game, probably because people do at most a few iterations of dominated strategies. How many levels of iteration do you think people actually engage in?

(c) Extra credit: Stage the game with some friends, with the payoff such that the winner receives \$10.00. Can you estimate how many stages of elimination of dominated strategies people go through?

2.17 Poker with Bluffing

Ollie and Stan decide to play the following game of poker. Each has a deck consisting of three cards, labeled H (high), M (medium) and L (low). Each puts \$1 in the pot chooses a card randomly from his deck and does not show the card to his friend. Ollie (player one) either stays, leaving the pot unchanged, or raises, adding \$1 to the pot. Stan simultaneously makes the same decision. If they both raise or they both stay, the player with the higher card wins the pot (which contains \$2 if they stayed and \$4 if they raised), and if they tie, they just take their money back. If Ollie raises and Stan stays, then Ollie gets the \$3 pot. However, if Stan raised and Ollie stayed, Ollie gets another chance. He can either drop, in which case Stan wins the \$3 pot (only \$1 of which is Ollie's), or he can call, adding \$1 to the pot. Then, as before, the player with the higher card wins the pot, and with the same card, they take their money back. A game tree for Poker with Bluffing is depicted in Figure 2.4 (the '?' in the figure means that the payoff depends on who has the higher card).

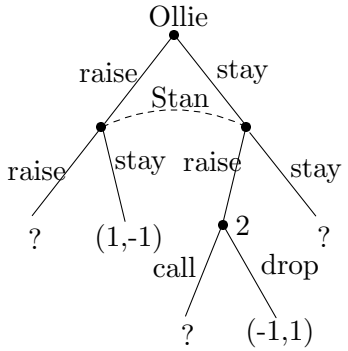


Figure 2.4: Poker with Bluffing

(a) Show that Ollie has 64 pure strategies and Stan has eight pure strategies;

(b)* Find the normal form game. Notice that while Poker with Bluffing is a lot simpler than real poker, it is nevertheless extremely laborious to find the normal form game for the players, since it is a 64×8

matrix! The only way to do this efficiently is to write a computer program. If you like programming, however, it is not a hard task.

(c)* Show that when you eliminate dominated strategies there are only nine pure strategies left for Ollie, and seven for Stan.

(d)* Show that Ollie always raises with H or M on the second round, and drops with L.

(e)* Suppose Stan sees Ollie's first move before deciding to stay or raise (i.e., the two nodes where Stan moves are separate information sets). Now find the normal form game. Notice that this matrix is 64×64, so calculating this by hand is quite prohibitive.

(f)* Eliminate dominated strategies, and show that Ollie has only ten strategies left, while Stan has twelve.

(g)* Show that Ollie always calls with a high card on the second round and Stan always raises with a high card.

2.18 The Centipede Game

Use backward induction to find a Nash equilibrium of the following game, which is called 'Rosenthal's Centipede Game.' The players, Mutt and Jeff, start out with \$1 in each, and they alternate rounds. On the first round, Mutt can defect (D) by stealing \$2 from Jeff and the game is over. Otherwise Mutt cooperates (C) by not stealing, and Nature gives Mutt \$1. Then Jeff can defect (D) and steal \$2 from Mutt, and the game is over, or he can cooperate (C), and Nature gives Jeff \$1. This continues until one or the other defects, or each player has \$100. The game tree is illustrated in Figure 2.5. Show that the only Nash equilibrium compatible with backward induction has Mutt defecting on the first round.

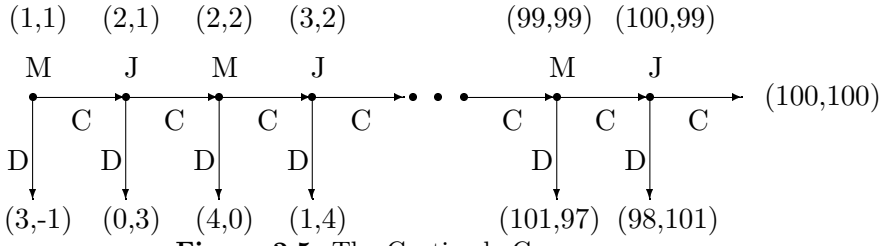


Figure 2.5: The Centipede Game

Now of course common sense tells you that this is not the way real players would act in this situation, and empirical evidence corroborates this intuition (McKelvey and Palfrey 1992). This issue surely merits further exploration, which we take up in Chapter 5.